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Application of He's variational iteration method for solving the reaction–diffusion equation with ecological parameters

D.D. Ganji[a], G.A. Afrouzi[b,*], R.A. Talarposhti[b]

[a] Department of Mechanical Engineering, Mazandaran University, Babol, Iran
[b] Department of Mathematics, Faculty of Basic Sciences, Mazandaran University, Babolsar, Iran

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Abstract

He's variational iteration method is applied to the search for the solution of the reaction–diffusion problem with ecological parameters. The method is extremely simple and concise, and comparison with the Adomian method reveals that the present method is an attracting mathematical tool.

Keywords: Variational iteration method (VIM); Adomian decomposition method (ADM)

1. Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. It is often useful for engineers and others to have an approximate closed form solution to describe a nonlinear problem. Although it is very easy for us now to find the solutions of some problems by means of computers, it is still rather difficult to solve nonlinear problems either numerically or theoretically. Also it is very difficult for us to obtain the exact solution for these problems. In recent decades, numerical analysis [1] and the approximate methods have been developed considerably for nonlinear partial equations.

Many different new methods have been presented recently such as the variational iteration method (VIM) by He [3–10] and the Adomian's decomposition method (ADM) [2]. VIM has many merits over classical approximate techniques, it can solve nonlinear equations easily and accurately. In this method, general Lagrange multipliers are introduced to construct correction functionals for the problems. The multipliers can be identified optimally via the variational theory. This method has recently been applied to various engineering problems [11–20].

In this paper, our purpose is to solve the reaction–diffusion equation with ecological parameters using He's variational iteration method and to compare the results with exact solutions or approximated analytic solutions obtained by ADM.

* Corresponding author. Tel.: +98 11252 42003; fax: +98 11252 42003.
E-mail address: afrouzi@umz.ac.ir (G.A. Afrouzi).

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Consider the nonlinear initial-boundary value parabolic problems:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + au\left(1 - \dfrac{u}{N}\right) & (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega, \end{cases} \tag{1}$$

where: $\Delta u = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $D > 0$ is the diffusion coefficient, $a > 0$ is the linear reproduction rate and $N > 0$ is the carrying capacity of the environment and Ω is a region in 3D space (see Murry [21] for details).

2. Adomian method for reaction–diffusion equation

The Adomian decomposition method provides an efficient and computationally convenient method for generating approximate series solutions to a wide class of equations.

We consider problem (1) in a 3D space (e.g. $\Omega = [0,1] \times [0,1] \times [0,1]$) and $T = 1$ and therefore, for the real value of $u(x,y,z,t)$, the reaction diffusion equation has the following form:

$$u_t = D\Delta u + au\left(1 - \frac{u}{N}\right), \tag{2}$$

with the initial and boundary conditions of

$$\begin{cases} u(x,y,z,t) = 0 & (x,y,z,t) \in \partial\Omega \times (0,T) \\ u(x,y,z,0) = u_0(x,y,z) \geq 0 & (x,y,z) \in \Omega. \end{cases} \tag{3}$$

Eq. (2) can be written as

$$L_t(u) = D(L_x(u) + L_y(u) + L_z(u)) + F(u), \tag{4}$$

where: $L_t = \frac{\partial}{\partial t}$, $L_x = \frac{\partial^2}{\partial x^2}$, $L_y = \frac{\partial^2}{\partial y^2}$, $L_z = \frac{\partial^2}{\partial z^2}$ and $F(u) = au\left(1 - \frac{u}{N}\right)$. If we operate the two sides of (4) with inverse operator of L_t, we have

$$u(x,y,z,t) = u(x,y,z,0) + DL_t^{-1}(L_x(u) + L_y(u) + L_z(u)) + L_t^{-1}F(u).$$

Substituting the initial condition of (3) in the last formula we have

$$u(x,y,z,t) = u_0(x,y,z) + DL_t^{-1}(L_x(u) + L_y(u) + L_z(u)) + L_t^{-1}F(u). \tag{5}$$

To solve problems (2) and (3) using ADM, we decompose the unknown function $u(x,y,z,t)$ by a series of components defined by

$$u(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) \tag{6}$$

and a series of Adomian's polynomials of nonlinear term $F(u)$ such as

$$F(u) = \sum_{n=0}^{\infty} A_n(x,y,z,t). \tag{7}$$

The polynomials A_n are given by the following general algorithms

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} F(u_\lambda)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots \tag{8}$$

where we write $u_\lambda = \sum_{i=0}^{\infty} \lambda^i u_i$.

For example:

$$\begin{cases} A_0 = F(u_0), \\ A_1 = u_1 F'(u_0), \\ A_2 = u_2 F'(u_0) + \dfrac{1}{2} u_1^2 F''(u_0), \\ A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \dfrac{1}{6} u_1^3 F(u_0), \\ \vdots \end{cases} \tag{9}$$

The remaining components $u_n(x, y, z, t)$ are calculated as follows:

$$\begin{cases} u_1 = \displaystyle\int_0^t (D\Delta u_0 + A_0)\mathrm{d}t, \\ u_2 = \displaystyle\int_0^t (D\Delta u_1 + A_1)\mathrm{d}t, \\ \vdots \\ u_n = \displaystyle\int_0^t (D\Delta u_{n-1} + A_{n-1})\mathrm{d}t. \end{cases} \tag{10}$$

So, we can calculate the terms of $u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t)$, which is approximated by $\phi_m[u] = \sum_{n=0}^{\infty} u_n$. We can choose m such that the approximate $\phi_m[u]$ has the desired accuracy, and by having more terms, we will have higher accuracy.

3. He's variational iteration method for the reaction–diffusion equation

To clarify the basic ideas of He's variational iteration method, we consider the following differential equation:

$$Lu + Fu = g(t), \tag{11}$$

where L is a linear operator, F a nonlinear operator and $g(t)$ a heterogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + F\tilde{u}_n(\tau) - g(t))\mathrm{d}\tau, \tag{12}$$

where λ is a general Lagrangian multiplier [3–10] which can be identified optimally via the variational theory.

Now, we want to solve the reaction–diffusion equation with VIM. According to VIM, we can write down the correction functional for Eq. (4)

$$\begin{aligned} u_{n+1}(t) = u_n(t) + \int_0^t \lambda \Big[& L_\tau u_n(x, y, z, \tau) - D(L_x u_n(x, y, z, \tau) + L_y u_n(x, y, z, \tau) \\ & + L_z u_n(x, y, z, \tau)) - a u_n(x, y, z, \tau) \left(1 - \frac{u_n(x, y, z, \tau)}{N}\right)\Big] \mathrm{d}\tau. \end{aligned} \tag{13}$$

The Lagrange multiplier is determined as $\lambda = -1$.

4. Examples

4.1. Example 1

Consider the nonlinear parabolic problems (2) and (3) with an initial condition of

$$u(x, y, z, 0) = u_0(x, y, z) = xy + yz + xz \geq 0 \quad \text{on } [0, 1] \times [0, 1] \times [0, 1]. \tag{14}$$

ADM method:

$$\begin{aligned}
u_0(x,y,z,t) &= xy+yz+xz,\\
u_1(x,y,z,t) &= a(xy+yz+xz)\left(1-\frac{1}{N}(xy+yz+xz)\right)t,\\
u_2(x,y,z,t) &= \frac{1}{2}D\left(-\frac{2}{N}a(y+z)^2-\frac{2}{N}a(x+z)^2-\frac{2}{N}a(y+x)^2\right)t^2\\
&\quad +a(xy+xz+yz)\left(1-\frac{1}{N}(xy+xz+yz)\right)t,
\end{aligned} \tag{15}$$

and so on, the solution $u(x,y,z,t)$ in the series form is given as

$$u(x,y,z,t)=\sum_{n=0}^{\infty}u_n(x,y,z,t). \tag{16}$$

VIM method:

$$\begin{aligned}
u_0(x,y,z,t) &= xy+yz+xz,\\
u_1(x,y,z,t) &= -\frac{1}{N}(xyN-yzN-xzN-atxyN+atx^2y^2+2atxy^2z+2atx^2yz-atyzN+aty^2z^2\\
&\quad +2atyz^2x-atxzN+atz^2x^2),\\
u_2(x,y,z,t) &= -\frac{1}{6N^3}(-6N^3atxy+6N^2atx^2y^2+12N^2atxy^2z\\
&\quad +12N^2atx^2yz-6N^3atyz+6N^2aty^2z^2+12N^2atyz^2x-6N^3atxz+6N^2atz^2x^2\\
&\quad -12a^3t^3x^3y^2Nz+4a^3t^3xy^2N^2z-12a^3t^3xy^3Nz^2-24a^3t^3x^2y^2Nz^2\\
&\quad +2a^3t^3x^4y^4+2a^3t^3y^4z^4+2a^3t^3z^4x^4+4a^3t^3x^2yN^2z-12a^3t^3x^3yNz^2-12a^3t^3y^2z^3Nx\\
&\quad +4a^3t^3yz^2N^2x-12a^3t^3yz^3Nx^2+2a^3t^3x^2y^2N^2-4a^3t^3x^3y^3N+8a^3t^3x^3y^4z\\
&\quad +8a^3t^3x^4y^3z+12a^3t^3x^2y^4z^2+24a^3t^3x^3y^3z^2+12a^3t^3x^4y^2z^2+12a^3t^3y^2z^3Nx\\
&\quad +4a^3t^3yz^2N^2x+8a^3t^3xy^4z^3+24x^2y^3z^3a^3t^3+24a^3t^3x^3y^2z^3+8a^3t^3x^4yz^3\\
&\quad +2a^3t^3y^2z^2N^2+4a^3t^3y^3z^3N+8a^3t^3y^3z^4x+12a^3t^3y^2z^4x^2+8a^3t^3yz^4x^3\\
&\quad +2z^2x^2N^2-4a^3t^3z^3x^3N+9a^2N^2t^2y^2z^2-18a^2Nt^2y^2z^3x-18a^2Nt^2yz^3x^2\\
&\quad +9a^2N^2t^2z^2x^2+9a^2t^2N^2x^2y^2-18a^2Nt^2x^2y^3z-18a^2Nt^2x^3y^2z\\
&\quad -18a^2Nt^2xy^3z^2-36a^2Nt^2x^2y^2z^2-18a^2Nt^2x^3yz^2-3a^2N^3t^2(xy+xz+yz)\\
&\quad +18a^2N^2t^2(x^2yz+yz^2x+xy^2z)-6a^2Nt^2y^3z^3-6a^2Nt^2x^3y^3\\
&\quad +12aN^2t^2D(y^2+z^2+x^2)+12aN^2t^2(yz+xz+xy)-6a^2Nt^2z^3x^3+6N^3yz\\
&\quad -6N^3xz-12a^3t^3x^2y^3Nz-6N^3xy). \quad \text{(see Figs. 1 and 2)}
\end{aligned} \tag{17}$$

4.2. Example 2

Consider the nonlinear parabolic problem (2) and (3) with an initial condition of

$$u(x,y,z,0)=u_0(x,y,z)=xy-z\geq 0 \quad \text{on } [0,1]\times[0,1]\times[0,1]. \tag{18}$$

ADM method:

$$\begin{aligned}
u_0(x,y,z,t) &= xy-z,\\
u_1(x,y,z,t) &= a(xy-z)\left(1-\frac{xy-z}{N}\right)t,\\
u_2(x,y,z,t) &= a(xy-z)\left(1-\frac{xy-z}{N}\right)t+\frac{1}{2}D\left(\frac{-2ay^2}{N}-\frac{-2ax^2}{N}-\frac{-2a}{N}\right)t^2,
\end{aligned} \tag{19}$$

and so on. The solution $u(x, y, z, t)$ in the series form is given as

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t). \tag{20}$$

VIM method (see Figs. 3 and 4):

$$\begin{aligned}
u_0(x, y, z, t) &= xy - z,\\
u_1(x, y, z, t) &= -\frac{1}{N}(-xyN + zN - atxyN + atx^2y^2 - 2atxyz + atzN + atz^2),\\
u_2(x, y, z, t) &= \frac{1}{6N^3}(2a^3t^3z^4 + 6N^3z - 6N^3atxy + 6N^2atx^2y^2 - 12N^2atxyz\\
&\quad + 12a^3t^3x^2y^2zN - 4a^3t^3xyN^2z - 12a^3t^3xyNz^2 + 2a^3t^3x^2y^2N^2 - 4a^3t^3x^3y^3N\\
&\quad - 8a^3t^3x^3y^3z + 12a^3t^3x^2y^2z^2 - 8a^3t^3xyz^3 + 6aN^2t^2D(y^2 + x^2)\\
&\quad + 9a^2N^2t^2x^2y^2 + 6a^2Nt^2x^2y^3 + 18a^2Nt^2x^2y^2z - 18a^2N^2t^2xyz - 18a^2Nt^2xyz^2\\
&\quad - 3a^2N^3t^2y + 6N^3atz + 6N^2atz^2 + 2a^3t^3x^4y^4 + 2a^3t^3z^2N^2 + 4a^3t^3z^3N + 6aN^2t^2D\\
&\quad + 9a^2N^2t^2z^2 + 6a^2Nt^2z^3 + 3a^2N^3t^2z - 6N^3xy).
\end{aligned} \tag{21}$$

4.3. Example 3

Consider the nonlinear parabolic problem (2) and (3) with an initial condition of

$$u(x, y, z, 0) = u_0(x, y, z) = xyz \geq 0 \quad \text{on } [0, 1] \times [0, 1] \times [0, 1]. \tag{22}$$

ADM method:

$$\begin{aligned}
u_0(x, y, z, t) &= xyz,\\
u_1(x, y, z, t) &= axyz\left(1 - \frac{xyz}{N}\right)t,\\
u_2(x, y, z, t) &= \frac{1}{2}\left(D\left(-\frac{2ay^2z^2}{N} - \frac{2ax^2z^2}{N} - \frac{2ay^2x^2}{N}\right)\right.\\
&\quad \left. + \; axyz\left(1 - \frac{xyz}{N}\right)\left(a\left(1 - \frac{xyz}{N}\right) - \frac{axyz}{N}\right)\right)t^2,
\end{aligned} \tag{23}$$

and so on. The solution $u(x, y, z, t)$ in the series form is given as

$$u(x, y, z, t) = \sum_{n=0}^{\infty} u_n(x, y, z, t). \tag{24}$$

VIM method (see Figs. 5 and 6):

$$\begin{aligned}
u_0(x, y, z, t) &= xyz,\\
u_1(x, y, z, t) &= -\frac{xyz\,(-N - atN + atxyz)}{N},\\
u_2(x, y, z, t) &= -\frac{1}{6N^3}(-6xyzN^3 - 6xyzN^3at + 6x^2y^2z^2N^2at - 3a^2t^2N^3xyz + 2a^3x^2y^2z^2t^3N^2\\
&\quad - 4a^3x^3y^3z^3t^3N + 2a^3x^4y^4z^4t^3 + 9a^2t^2N^2x^2y^2z^2 + 6at^2N^2D(y^2z^2 + x^2z^2 + x^2y^2)\\
&\quad - 6a^2t^2Nx^3y^3z^3).
\end{aligned} \tag{25}$$

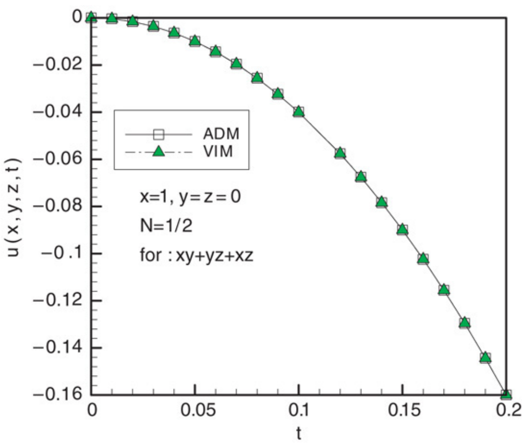

Fig. 1. The comparison of the results of two methods for $u(x, y, z, t)$ for the initial condition, at $u(x, y, z, 0) = u_0(x, y, z) = xy + yz + xz \geq 0$ when $N = \frac{1}{2}$, $x = 1$, $z = y = 0$.

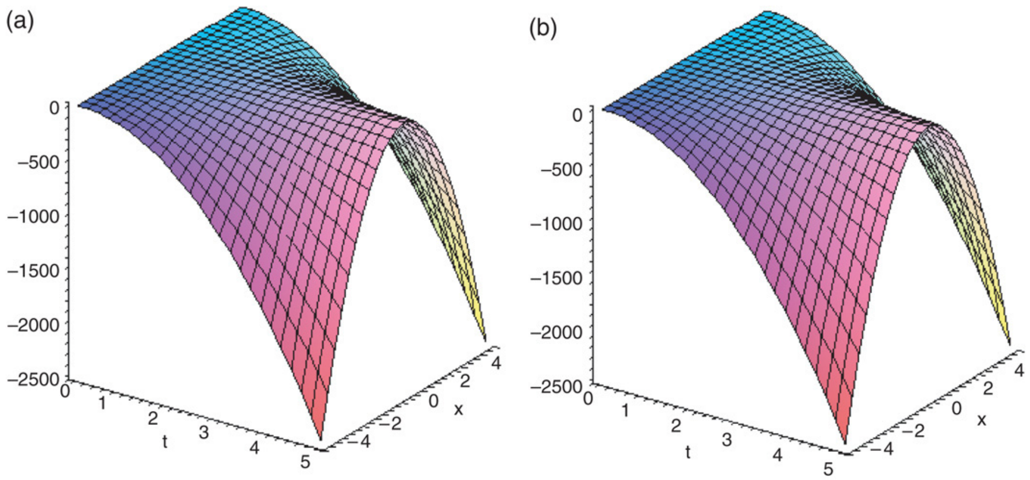

Fig. 2. (a) VIM and (b) ADM results for $u(x, y, z, 0) = u_0(x, y, z) = xy + yz + xz \geq 0$, for the solitary wave solution with the first initial condition (3) of Eq. (2).

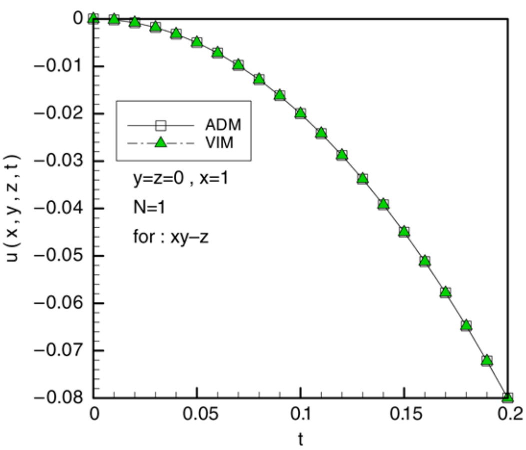

Fig. 3. The comparison of the results of the two methods for $u(x, y, z, t)$, for the initial condition, at $u(x, y, z, 0) = xy - z$ when $N = 1$, $x = 1$, $y = z = 0$.

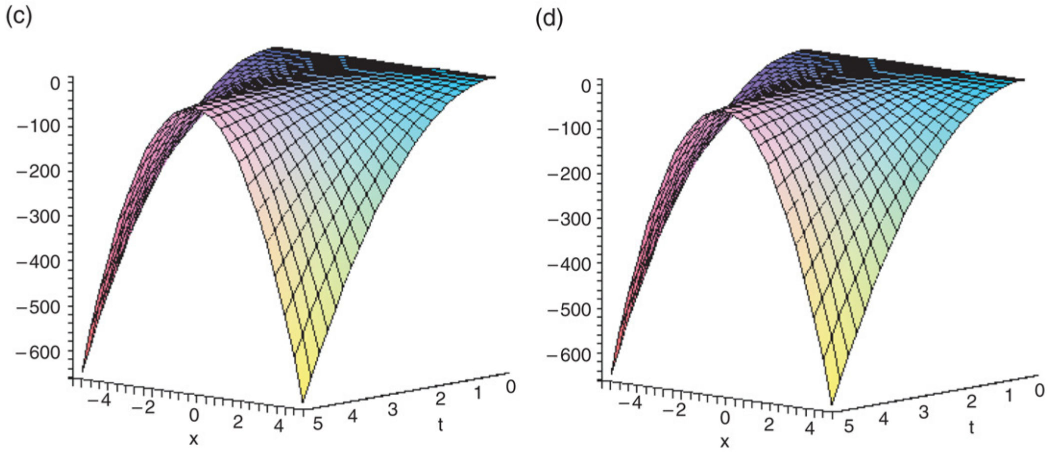

Fig. 4. (c) VIM and (d) ADM results for the solitary wave solution with the first initial condition (3) of Eq. (2), for $u(x, y, z, 0) = u_0(x, y, z) = xy - z \geq 0$.

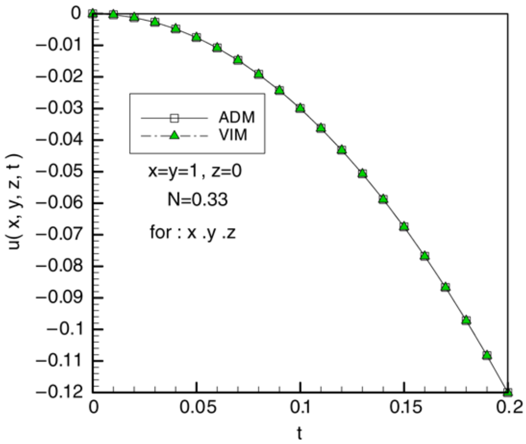

Fig. 5. The comparison of the results of the two methods for $u(x, y, z, t)$, for the initial condition, at $u(x, y, z, 0) = u_0(x, y, z) = xyz$ when $N = \frac{1}{3}x = y = 1, z = 0$.

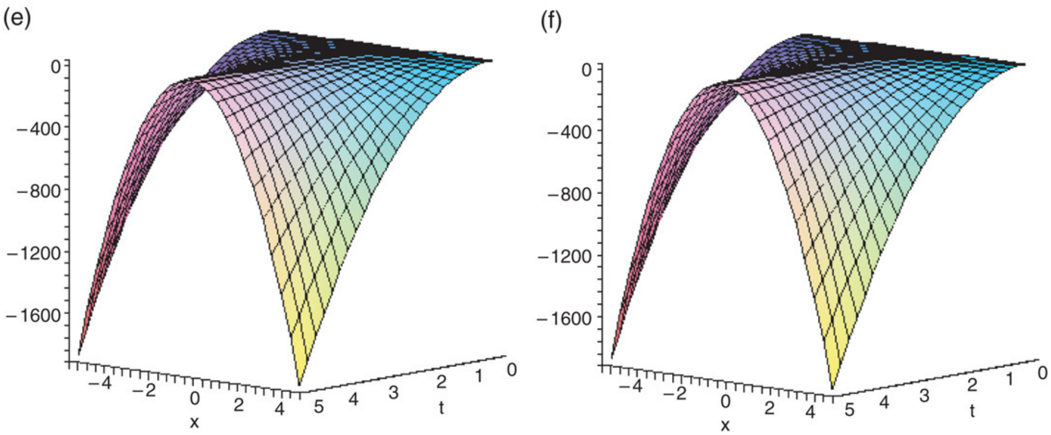

Fig. 6. (e) VIM and (f) ADM results for the solitary wave solution with the first initial condition (3) of Eq. (2), for $u(x, y, z, 0) = u_0(x, y, z) = xyz \geq 0$.

5. Conclusions

In this paper, He's variational iteration method has been successfully applied to find the solution of the nonlinear reaction–diffusion equation. All examples show that the results of the present method are in excellent agreement with those of ADM, and the obtained solutions are also shown graphically.

In our work, we use the Maple Package. Some of the advantages of the variational iteration method are that the initial solution can be freely chosen with some unknown parameters and that we can easily achieve the unknown parameters in the initial solution. An interesting point about VIM is that only few iterations or, even in some special cases, one iteration, lead to exact solutions or solutions with high accuracy.

The main merits of the variational iteration method are

1. VIM can overcome the difficulties arising in calculation of Adomian polynomials.
2. VIM does not require small parameters which are needed in perturbation method.
3. No linearization is needed; the method is very promising of finding wide application in nonlinear equations.

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